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COMMENT

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Abstract. The topological invariants of monpoles introduced by Taubes are described in a Chern-Weil framework.

Consider a non-Abelian monopole (A_j, Φ), with the Higgs field in the adjoint representation of a semisimple and compact Lie group G (for a review on monopoles see, e.g., Goddard and Olive (1978)). The expression

$$I = \int_{S^2} \mathrm{Tr}(F\Phi) \tag{1}$$

(where F is the field strength tensor and S^2 is the 2-sphere at infinity) is known to be a *topological invariant*, i.e. to depend only on $[\Phi] \in \pi_2(\mathrm{G/H})$, where H is the residual symmetry group, $\mathrm{G/H} = \mathrm{G}\Phi_0$. (1) appears, for example, in the Bogomolni bound for the energy.

(1) has been generalised by Taubes (1981) to

$$I^{(k)} = \int_{S^2} \mathrm{Tr}(F\Phi^k). \tag{2}$$

$[\Phi]$ is now a p-tuple of integers $m_1, \dots, m_p$, where p is the dimension of the centre of the Lie algebra $\mathfrak{h}$ of H. Taubes (1981) has proved that (2) is a linear combination of the m_j (see also O'Sé *et al* 1984).

Recently (Horváthy and Rawnsley 1985) (2) has been further generalised. We have shown that, if f is an arbitrary invariant function on $\mathfrak{G} \times \mathrm{G/H}$ which is linear in the first variable and if Φ transforms according to an *arbitrary* representation of G, then

$$I^{(f)} = \int_{S^2} f(F, \Phi) \tag{3}$$

is a topological invariant as long as Φ is covariantly constant. As a matter of fact, the value of (3) depends only on the free part of $[\Phi]$ and is easily calculated (see (8)-(10) below). Such expressions are needed to derive generalised Dirac conditions (Goddard and Olive 1978, 1981, Horváthy and Rawnsley 1985).

The proof is based on constructing an isomorphism ρ between the free part of $\pi_1(\mathrm{H})$ and a p-dimensional lattice in the centre of $\mathfrak{h}$. Here we give first a new Chern-Weil type construction§ to this map ρ. Next, we use the new framework to rederive the above results on generalised invariants.

§ We understand that there is a related result by Straumann and Wipf (1985).

In monopole theory one starts with a trivial 'unifying' bundle $Q=\mathbb{R}^3\times G$, where G—a compact and simply connected Lie group—is the 'unifying group'. At large distances the G symmetry is spontaneously broken to a closed subgroup H of G. Geometrically, over $S^2(Q, G)$ is reduced to a principal H bundle P. Any such reduction is produced by an equivariant 'reducing map' (Kobayashi and Nomizu 1963). Choosing a global trivialisation of Q, the reducing map can be identified with a map $\Phi: S^2\to G/H$—the physical Higgs field. Φ defines a homotopy class $[\Phi]\in\pi_2(G/H)$. Let $\delta: \pi_2(G/H)\to\pi_1(H)$ denote the connecting homomorphism. Two monopole bundles are isomorphic iff the corresponding classes in $\pi_2(G/H)$ (or in $\pi_1(H)$) are the same. $\pi_1(H)=\pi_1(H)_{\text{free}}+\pi_1(H_{ss})$, where $\pi_1(H)_{\text{free}}=Z^p$, p being the dimension of $Z(\mathfrak{h})$. H_{ss} is the semisimple subgroup of H generated by the derived algebra $[\mathfrak{h},\mathfrak{h}]$. $\pi_1(H_{ss})$ is a finite Abelian group.

The free part is described as follows: denote by $\Gamma=\{\xi\in\mathfrak{h}\,|\exp 2\pi\xi=1\}$ the unit lattice of H, and let $z:\mathfrak{h}\to Z(\mathfrak{h})$ be the projection of the Lie algebra of H onto its centre. $z(\Gamma)$, the image of Γ under z, is a lattice in $Z(\mathfrak{h})$, whose dimension is the same as that of $Z(\mathfrak{h})$, say ρ. In Horváthy and Rawnsley (1985) we have proved the following theorem. Define, for any loop γ in H,

$$\rho(\gamma)=(1/2\pi)\int_\gamma z(\theta_H)\in Z(\mathfrak{h}) \tag{4}$$

where $\theta_H=g^{-1}\,dg$ is the canonical (Maurer-Cartan) 1-form of H. ρ defines an isomorphism of $\pi_1(H)_{\text{free}}$ with $z(\Gamma)$. If $\zeta_1,\ldots,\zeta_p$ is a $\mathbb{Z}$ basis for $z(\Gamma)$, then $\rho(\gamma)=\Sigma m_i\zeta_i$ provides us with p 'quantum' numbers $m_1,\ldots,m_p$. Any loop in H is known to be homotopic to one of the form $\gamma(t)=\exp(2\pi\xi t)$, for which $\rho(\gamma)=z(\xi)$.

$\rho(\Phi)=\rho(\delta[\Phi])$ can also be calculated as the integral of a 2-form over the 2-sphere at infinity:

$$\rho(\Phi)=(1/2\pi)\int_{S^2}\Phi^*\Omega \tag{5}$$

where Ω is the projection of G/H of the $Z(\mathfrak{h})$-valued 2-form $z(d\theta_G)$ on G. The situation can also be understood from a Chern-Weil-type viewpoint (Kobayashi and Nomizu 1969). Let us consider an arbitrary connection form A on P, and denote by $F=DA$ its curvature form. $z(F)$ is a $Z(\mathfrak{h})$-valued 2-form on P, which is horizontal and basic, since

$$r_g^*z(F)=z(r_g^*F)=z(A\,dg^{-1}F)=z(F)$$

so $z(F)$ descends to S^2 to a $Z(\mathfrak{h})$-valued 2-form Ω^A, $z(F)=\pi^*\Omega^A$. This 2-form is closed:

$$d(z(F))=z(dF)=z(DF-[A,F])=z(DF)=0$$

since z vanishes on the derived algebra and $DF=0$ by the Bianchi identity.

Let A' be a second connection form on P and consider $B=A'-A$. B is a basic 1-form of the adjoint type so

$$F'=dA'+\tfrac{1}{2}[A'\wedge A']=F+D^AB+\tfrac{1}{2}[B\wedge B].$$

Thus

$$z(F')=z(F)+z(D^AB)=z(F)+d(z(B)).$$

But $z(B)$ is an invariant horizontal 1-form which descends to a 1-form β on S^2, and hence $\Omega^{A'}=\Omega^A+d\beta$. This proves the following theorem.

Theorem. The cohomology class $[\Omega^A] \in H^2\, dR(S^2) \otimes Z(\mathfrak{h})$ is independent of the choice of the connection A on P. Consequently, the integral

$$\rho(P) = (1/2\pi) \int_{S^2} z(F) \in Z(\mathfrak{h}) \tag{6}$$

depends only on the bundle P.

Let us now assume that (P, H) is the reduction of the trivial unifying bundle (Q, G) defined by Φ: $S^2 \to \mathrm{G}/\mathrm{H}$ and so can be identified with the pullback by Φ of G, viewed as a principal H bundle over G/H. The $\mathfrak{h}$ component of θ_{G} defines a connection on the principal H bundle G whose pullback by Φ is a connection form A on P. It is not difficult to show that $z(DA) = \Phi^*\Omega$, where Ω is the 2-form defined above. Thus we have established the equivalence of our new construction with those given before.

Monopole fields also satisfy the Yang–Mills–Higgs equations. Assuming a sufficiently rapid fall-off at infinity, on S^2 the Yang–Mills–Higgs equation reduces to $D^*F = 0$, where the * is the Hodge operator on the Riemannian manifold S^2. The solution of this equation has been given by Goddard *et al* (1977). There exists a constant vector Π in $\mathfrak{h}$ such that $F = \mathcal{F}\Pi$, where $\mathcal{F}$ is the canonical area form of the 2-sphere. Π is quantised, $\exp(4\pi\Pi) = 1$. The vector Π can be chosen without loss of generality in any given Cartan subalgebra of $\mathfrak{h}$.

This theorem can also be reformulated (Friedrich and Habermann 1985) as the following theorem.

Theorem. The holonomy group of asymptotic monopole bundles is a U(1), generated by the 'non-Abelian charge' vector Π.

Here we give a new proof for this statement. The field strength tensor F is a section of the bundle of Lie algebra-valued 2-forms on S^2, so $F = (\pi^*\mathcal{F})\psi$, where π is the projection $P \to S^2$, the * denotes pullback and ψ is an adjoint Higgs-type field, $\psi(pg) = A\, dg^{-1}\psi(p)$. $^*F = \psi$ (Hodge star) and hence the asymptotic field equation reduces to $D\psi = 0$: ψ is a covariantly constant Higgs field.

If X and Y are vector fields on P, then $F(X, Y) = f\psi$, where f is the real function $f = \pi^*F(X, Y)$. Thus for any horizontal vector field Z on P:

$$Z(F(X, Y)) = Z(f)\psi - f[A(Z), \psi] = Z(f)\psi$$

because $0 = D\psi = d\psi + [A, \psi]$ and Z is horizontal. By iteration we get

$$Z_1(\ldots Z_k\{F(X, Y)\}\ldots) = f^{(k)}\psi$$

for any horizontal vector fields Z_j, $j = 1, \ldots, k$. We conclude that $Z(F(X, Y))$ is parallel to ψ. Ozeki's theorem (Kobayashi and Nomizu 1963, p 101) tells us, however, that the infinitesimal holonomy is generated by the expressions of this form. The infinitesimal holomy is thus one dimensional, generated by a constant vector Π in the Lie algebra. Finally, if the bundle is non-trivial, the holonomy group must be a U(1), rather than merely $\mathbb{R}$, showing that Π must be quantised.

The transition function of a monopole is now homotopic to $h(t) = \exp(4\pi\Pi t)$, $0 \leq t \leq 1$. $\rho(P)$ is thus simply $\rho(P) = 2z(\Pi)$. Let us now consider an invariant function f on $\mathfrak{G} \times (\mathrm{G}/\mathrm{H})$ which is linear in the first variable. Such a function can be viewed alternatively as an equivariant map f: $\mathrm{G}/\mathrm{H} \to \mathfrak{G}^*$ (algebraic dual of $\mathfrak{G}$), characterised by $f_0 = f(eH) \in \mathfrak{G}^*$. So invariant functions correspond to H-invariant elements in the dual, i.e. to elements in $(\mathfrak{G}^*)^{\mathrm{H}}$. Let us consider the real 2-form $\langle f\circ\Phi, F\rangle$ on P. (The bracket $\langle\ ,\ \rangle$ here denotes the pairing between $\mathfrak{G}^*$ and $\mathfrak{G}$.) It is closed, $\mathrm{d}\langle f\circ\Phi, F\rangle = \langle D(f\circ\Phi), F\rangle + \langle f\circ\Phi, DF\rangle = 0$, since f is linear in the first variable and $D\Phi=0$, and by the Bianchi identity. It is also horizontal, since F is horizontal. So it descends to a unique closed real 2-form Ω^f on M:

$$\langle f\circ\Phi, F\rangle = \pi^*\Omega^f. \tag{7}$$

Proposition. The cohomology class $[\Omega^f]$ is independent of the connection A and depends only on the bundle (P, H).

Indeed, the image of $\chi = f\circ\Phi$ is a coadjoint orbit G/G_0, where G_0 is the stability group of f_0. $D\chi=0$ since f is linear. Hence (P, A) reduces to the G_0 bundle P_0 with connection A_0. $P_0=\{p\in P\,|\,\chi(p)=f_0\}$ (Kobayashi and Nomizu 1963) and therefore $\langle\chi, F\rangle = \langle f\circ\Phi, F\rangle = \langle f_0, F\rangle$ on P_0.

The point is that f_0 is a first-order invariant function on $\mathfrak{G}_0$ (the Lie algebra of G_0) and so (7) is a Chern–Weil form representing the bundle P_0, independent of the choice of A_0, by the Chern–Weil theorem (Kobayashi and Nomizu 1969). Summarising, we get the following.

Theorem. For any invariant function f on $\mathfrak{G}\times(\mathrm{G}/\mathrm{H})$ the integral (3) depends only on the homotopy class $[P]\in\pi_1(\mathrm{H})$, provided that the reducing map Φ is covariantly constant. Its value is calculated by

$$I^{(f)} = \int_{S^2}\Omega^f = 2\pi\langle f_0, \rho(P)\rangle. \tag{8}$$

Proof. The integrand in (3) is, by (7), just Ω^f. But the cohomology class $[\Omega^f]$ depends only on the homotopy class $[P]\in\pi_1(\mathrm{H})$. Hence, so also does the integral (3). Observe finally that if $f_0\in(\mathfrak{G}^*)^{\mathrm{H}}$, then

$$f_0(\eta) = f_0(z(\eta)) \qquad \text{for} \qquad \eta\in\mathfrak{h}.$$

But F is $\mathfrak{h}$-valued and hence $\langle f\circ\Phi, F\rangle = \langle f_0, F\rangle = \langle f_0, z(F)\rangle$. Consequently

$$I^f = \int_{S^2}\Omega^f = \left\langle f_0, \int_{S^2} z(F)\right\rangle = 2\pi\langle f_0, \rho(P)\rangle.$$

For a monpole bundle defined by a non-Abelian charge vector Π, this simplifies further to

$$I^f = 4\pi\langle f_0, \Pi\rangle. \tag{9}$$

Indeed, for a monpole $D\Phi=0$ exactly when $\Pi\in\mathfrak{h}$, and then $\rho(P)=2z(\Pi)$. Assume, in particular, that G is semisimple and simply connected and φ is in the adjoint representation. Choose a Cartan subalgebra T, let α_i, $i=1,\ldots,r=\operatorname{rank}\mathrm{G}$ be the simple roots and define η_α for a root α by $\mathrm{Tr}(\eta_\alpha\xi)=\alpha(\xi)$, $\xi\in T$. Define the indices $i_j 1, j=1,\ldots,p$ by $\alpha_{i_j}(\varphi_0)=0$. Denote $\eta_i = 2\eta_{\alpha_i}/\alpha_i(\eta_{\alpha_i})$. Then $z(\eta_{i_j}), j=1,\ldots,p$ form a $\mathbb{Z}$ basis for $z(\Gamma)$, so $\rho[\varphi]=\Sigma_j m_j z(\eta_{i_j})$. f_0 is linear, so

$$I^{(f)} = 2\pi\sum_{j=1}^{p}\langle f_0, \eta_{i_j}\rangle m_j. \tag{10}$$

For $f(F,\varphi)=\mathrm{Tr}(F,\varphi^k)$ we have $\langle f_0,\xi\rangle = \mathrm{Tr}(\varphi_0^k\xi)$ and (10) reduces to (4.26) and (4.27) in Horváthy and Rawnsley (1984).

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